ON THE TUNNEL NUMBER AND THE MORSE-NOVIKOV NUMBER OF KNOTS

A.V.PAJITNOV

ABSTRACT. Let L be a link in S^3; denote by $\mathcal{MN}(L)$ the Morse-Novikov number of L and by $t(L)$ the tunnel number of L. We prove that $\mathcal{MN}(L) \leqslant 2t(L)$ and deduce several corollaries.

1. INTRODUCTION

1.1. **Background.** Let L be a link in S^3, that is, an embedding of several copies of S^1 to S^3. First off, we recall the definition of three numerical invariants of L. In the sequel $N(L)$ denotes a closed tubular neighbourhood of L.

A. Tunnel Number. An arc γ in S^3 is called *a tunnel* for L if $\gamma \cap L$ consists of the two endpoints of γ. The tunnel number $t(L)$ is the minimal number m of disjoint tunnels $\gamma_1, \ldots, \gamma_m$ such that the closure of $S^3 \setminus N(L \cup \gamma_1 \cup \ldots \cup \gamma_m)$ is a handlebody. The tunnel number was introduced by B. Clark in [1]; this invariant was studied in the works of K. Morimoto, M.Sakuma, Y. Yokota, T. Kobayashi, M. Scharlemenn, J. Schultens and others (see [11], [9], [10], [14]).

For any two knots K_1, K_2 we have $t(K_1 \# K_2) \leqslant t(K_1) + t(K_2) + 1$. In the paper [8] T.Kobayashi and Y. Rieck defined the growth rate for a knot K by the formula

$$gr_t(K) = \limsup_{m \to \infty} \frac{t(mK) - mt(K)}{m - 1}$$

where mK stands for the connected sum of m copies of the knot K. We have $1 \geqslant gr_t(K) \geqslant -t(K)$.

B. Bridge numbers. Let $S^3 = H_1 \cup H_2$ be a Heegaard decomposition of S^3; put $\Sigma = H_1 \cap H_2$, and $g = g(\Sigma)$. We say (following H. Doll [2]) that L is in a n-bridge position with respect to Σ if Σ intersects L in $2n$ points and $\Sigma \cap H_i$ is a union of n trivial arcs in H_i for $i = 1, 2$. The g-bridge number $b_g(L)$ of L is defiined as the minimal number n such that L can be

put in a n-bridge position with respect to a Heegaard decomposition of genus g (thus $b_0(L)$ is the classical bridge number as defined n the paper [15] of H. Schubert). We have

$$t(L) \leqslant g + b_g(L) - 1.$$

C. Morse-Novikov numbers. A *framing* of L is a diffeomorphism $\phi : L \times D^2 \to N(L)$. Let C_L denote the closure of $S^3 \setminus N(L)$. A Morse function $f : C_L \to S^1$ is called *regular* if its restriction to the boundary $\partial N(L)$ satisfies the following relation: $(f \circ \phi)(l, z) = \frac{z}{|z|}$. A regular Morse function has finite number of critical points; the number of the critical points of f of index i wil be denoted by $m_i(f)$; the total number of critical points of f will be denoted by $m(f)$. The minimal value of $m(f)$ over all possible framings ϕ and all possible Morse maps $f : C_L \to S^1$ is called *the Morse-Novikov number of the link* L and denoted by $\mathcal{MN}(L)$ (see [12]). The Morse-Novikov theory implies that

$$\mathcal{MN}(L) \geqslant 2(b_1(L) + q_1(L))$$

where $b_1(L)$ and $q_1(L)$ are the *Novikov numbers* defined as follows. Let $\bar{C}_L$ be the infinite cyclic covering induced by f from the covering $\mathbf{R} \to S^1$. Denote the ring $\mathbf{Z}[t, t^{-1}]$ by Λ, and the ring $\mathbf{Z}((t))$ by $\widehat{\Lambda}$. Then $b_1(L)$ and $q_1(L)$ are respectively the rank and torsion numbers of the module $H_1(\bar{C}_L) \underset{\Lambda}{\otimes} \widehat{\Lambda}$. In case when the Novikov numbers are not sufficient to determine the $\mathcal{MN}(L)$ the twisted Novikov numbers (introduced by H. Goda and the author in [3]) are useful.

As for the upper bounds for $\mathcal{MN}(L)$ not much is known. M. Hirasawa proved that for every 2-bridge knot K we have $\mathcal{MN}(K) \leqslant 2$ (unpublished). In the papers [13] and [6] of Lee Rudolph and M. Hirasawa it is proved that $\mathcal{MN}(K) \leqslant 4g_f(K)$ where $g_f(K)$ is the *free genus* of K, that is, the minimal possible genus of a Seifert surface Σ bounding K such that $S^3 \setminus \Sigma$ is an open handlebody.

1.2. **Main results.** The main result of this work is

Theorem 1.1. *For every link L in S^3 we have*

$$\mathcal{MN}(L) \leqslant 2t(L). \tag{1}$$

The following corollaries are easily deduced.

Corollary 1.2. *For every g we have*

$$\mathcal{MN}(L) \leqslant 2(g + b_g(L) - 1).$$

Corollary 1.3. *For every (1,1)-knot K we have $\mathcal{MN}(K) \leqslant 2$.*

Corollary 1.4. *For every link L we have*

$$q_1(L) + b_1(L) \leqslant t(L).$$

Corollary 1.5. *For every knot K*

$$gr_t(K) \geqslant -t(K) + q_1(K).$$

2. Proof of Theorem 1.1

Let $m = t(L)$. Pick a framing $\phi : L \times D^2 \to N(L)$. Then the manifold $C_L = \overline{S^3 \setminus N(L)}$ is obtained from ∂C_L by attaching m one-handles and then attaching a handlebody of genus $(m+1)$ to the resulting cobordism. Thus we obtain a Morse function $g : C_L \to \mathbf{R}$ which is constant on ∂C_L and has the following Morse numbers: $m_0(g) = 0,\ m_1(g) = m,\ m_2(g) = m+1,\ m_3(g) = 1$. Pick any Morse map $h : C_L \to S^1$ such that $h|\partial C_L$ is the canonical fibration: $(h \circ \phi)(l, z) = \frac{z}{|z|}$. Consider a closed 1-form $\omega_\epsilon = dg + \epsilon dh$. For $\epsilon > 0$ sufficiently small ω_ϵ is a Morse form with the same Morse numbers as dg. Therefore the form

$$\frac{1}{\epsilon}\omega_\epsilon = d\Big(\frac{1}{\epsilon} g + h\Big)$$

is the differential of a Morse map $g_1 : C_L \to S^1$ having the required behaviour on ∂C_L. The map g_1 has one local maximum, and the standard elimination procedure (see for example [12] for details) gives us a Morse function $f : C_L \to S^1$ with $m_0(f) = 0,\ m_1(f) = m,\ m_2(f) = m,\ m_3(f) = 0$. Thus $\mathcal{MN}(L) \leqslant 2m$.

3. Examples, and further remarks

A theorem of M. Hirasawa says that $\mathcal{MN}(K) \leqslant 2$ if K is a two-bridge knot. Since $t(K) \leqslant b(K) - 1$ our theorem implies this result. Observe that the proof of the M. Hirasawa's theorem uses the H. Schubert's classification of 2-bridge knots, and can not be generalized to the case of arbitrary bridge number.

The inequality (1) implies also the upper bound

$$\mathcal{MN}(K) \leqslant 4g_f(K)$$

obtained by Lee Rudolph and M. Hirasawa (see [13], [6]). Indeed Jung Hoon Lee [5] has shown that $t(K) \leqslant 2g_f(K)$.

In many cases the estimate of Theorem 1.1 is better than the free genus estimate. For example, for a pretzel knot $K = P(-2, m, n)$ where $m, n \geqslant 3$ are odd numbers, we have $g(K) = \frac{m+n}{2}$ (see [4]), so that $g_f(K) \geqslant \frac{m+n}{2}$. On the other hand $t(K) = 1$.

4. The tunnel number and the homology with local coefficients

Let L be a link in S^3, put $m = t(L)$. As we have observed in the previous section there is a Morse function $g : C_L \to \mathrm{R}$ such that g is constant on ∂C_L and takes there its minimal value, and with the Morse numbers as follows: $m_0(g) = 0$, $m_1(g) = m$, $m_2(g) = m+1$, $m_3(g) = 1$. The function $-g$ provides a handle decomposition of the manifold C_L, with $(m+1)$ one-handles, therefore we have the usual homological estimate $m + 1 \geqslant \mu_{\mathrm{Z}}(H_1(C_L, \mathrm{Z}))$. † For the case of knots this estimate is trivial, however in some cases we can improve it using homology with local coefficients. Let $\rho : \pi_1(C_L) \to GL(q, R)$ be a *right* representation (that is, $\rho(ab) = \rho(b)\rho(a)$ for every a, b). Denote by $\widetilde{C}_L$ the universal covering of C_L. The homology of the chain complex

$$C_*(\widetilde{C}_L, \rho) = R^q \underset{\rho}{\otimes} C_*(\widetilde{C}_L)$$

is called *the homology with local coefficients* ρ or ρ*-twisted homology* and denoted by $H_*(C_L, \rho)$. If R is the principal ideal domain, then we have

$$m + 1 \geqslant \frac{1}{q}\, \mu_R\Big(H_1(C_L, \rho)\Big). \tag{2}$$

In what follows we will concentrate on the case of knots. For a knot K consider a meridional embedding $i : S^1 \to C_K$. Given a right representation $\rho : \pi_1(C_K) \to GL(q, R)$ we can induce it to S^1 by i and obtain a local coefficient system $i^*\rho$ on S^1.

The following proposition is an easy corollary of the main theorem of the paper of D. Silver and S. Williams [16].

† For a finitely generated R-module T we denote by $\mu_R(T)$ the minimal number of generators of T.

Proposition 4.1. *Let K be any knot in S^3. Then there is a right representation $\gamma : \pi_1(C_K) \to GL(q, R)$ with R a principal ideal domain such that*

(i) $H_1(C_K, \gamma) \neq 0$,
(ii) $H_1(S^1, i^*\gamma) = 0$.

Proof. Let us first recall briefly the Silver-Williams theorem. Consider the meridional homomorphism $\xi : \pi_1(C_K) \to \mathbf{Z}$ as a homomorphism of $\pi_1(C_K)$ to $\Lambda^\bullet = GL(1, \Lambda)$, where $\Lambda = \mathbf{Z}[t, t^{-1}]$. For a right representation $\theta : \pi_1(C_K) \to GL(q, \mathbf{Z})$ form the tensor product $\rho = \xi \otimes \theta : \pi_1(C_K) \to GL(q, \Lambda)$. Consider the Λ-module $\mathfrak{B} = H_1(C_K, \rho)$ and choose a free resolution for $\mathfrak{B}$:

$$0 \longleftarrow \mathfrak{B} \longleftarrow \Lambda^r \xleftarrow{p} \Lambda^k \longleftarrow \cdots \tag{3}$$

where $k \geqslant r$. The GCD of the ideal of Λ generated by the $r \times r$-minors of p is called the *twisted Alexander polynomial* of k with respect to ρ; we will denote it $\Delta(K, \theta)$ (it is defined up to multiplication by $\pm t^i$). The Silver-Williams theorem says that for every K there is a representation θ such that $\Delta(K, \theta)$ is not a unit of Λ. Pick such a representation θ and consider two cases:

1) $\Delta(K, \theta)$ is a monomial, that is $\Delta(K, \theta) = at^n$ with $a \in \mathbf{Z}$, $a \neq \pm 1$. In this case define the representation $\gamma = \widehat{\rho}$ to be the composition of ρ with the natural inclusion $GL(q, \Lambda) \subset GL(q, \widehat{\Lambda})$. The $\widehat{\rho}$-twisted homology is

$$H_1(C_K, \widehat{\rho}) = H_1(C_K, \rho) \underset{\Lambda}{\otimes} \widehat{\Lambda}.$$

We can obtain a free $\widehat{\Lambda}$-resolution for this module by tensoring the resolution (3) by $\widehat{\Lambda}$ over Λ. Since the GCD of elements of Λ remains the same when we extend the ring Λ to $\widehat{\Lambda}$ (see [12], Lemma 2.3), the GCD of the $r \times r$-minors of the matrix $\widehat{p}$ equals a. Since $\widehat{\Lambda}$ is a principal ideal domain we deduce that $H_1(C_K, \widehat{\rho})$ is non-zero and moreover it contains a cyclic direct summand. The property (ii) is easy to check.

2) $\Delta(K, \theta)$ is a polynomial of non-zero degree. In this case consider the ring $\Lambda_{\mathbf{Q}} = \mathbf{Q}[t, t^{-1}]$. This ring is principal and $\Delta(K, \theta)$ is not invertible in it; define the representation $\gamma = \widetilde{\rho}$ to be the composition of ρ with the natural inclusion $GL(q, \Lambda) \subset$

$GL(q, \Lambda_{\mathbf{Q}})$. The same argument as for the point 1) works here as well. □

Proposition 4.2. *Let K be any knot in S^3. Then there is $\lambda > 0$ such that for every $n \in \mathbf{N}$ we have $t(nK) \geqslant n\lambda - 1$.*

Proof. Pick a representation $\gamma : \pi_1(C_K) \to GL(q, R)$ satisfying the conclusion of Proposition 4.1. The module $H_1(C_K, \rho)$ contains then a cyclic R-submodule T.

Lemma 4.3. *For any $n \geqslant 1$ there is a right representation $\gamma_n : \pi_1(C_{nK}) \to GL(q, R)$ such that the module $\mathfrak{B}_n = H_1(C_{nK}, \gamma_n)$ contains a submodule isomorphic to nT.*

Proof. We proceed by induction in n. Denote by $\mu \in \pi_1(C_K)$ the meridional element. Assume that we have constructed $\gamma_n : \pi_1(C_{nK}) \to GL(q, R)$ in such a way that $\gamma_n(\mu) = \gamma(\mu)$. The group $\pi_1(C_{(n+1)K})$ is isomorphic to the amalgamated product of the groups $\pi_1(C_K)$ and $\pi_1(C_{nK})$ over the subgroup $\mathbf{Z}$ included to both groups via the embedding of the meridian. Let $\gamma_{n+1} : \pi_1(C_{(n+1)K}) \to GL(q, R)$ be the product of the representations γ and γ_n. Using the property (ii) from 4.1 and the Mayer-Vietoris exact sequence it is easy to deduce that the module $\mathfrak{B}_{(n+1)} = H_1(C_{(n+1)K}, \gamma_{(n+1)})$ contains a submodule isomorphic to the direct sum of T and nT. □

The previous Lemma implies that $\mu_R(\mathfrak{B}_n) \geqslant n$. Our proposition follows, since

$$t(nK) + 1 \geqslant \frac{1}{q}\, \mu_R(\mathfrak{B}_n) \geqslant \frac{n}{q}.$$

□

Corollary 4.4. *For any knot K we have*

$$gr_t(K) > -t(K).$$

□

5. Generalizations and a question

The generalization of the results of the Section 1.2 to the case of knots and links in an arbitrary closed 3-manifold are straightforward. The same goes for the formula (2). On the other hand it is not clear at all whether the Proposition 4.1 and 4.2 admit such generalizations, since the analogs of Silver-Williams theorem for arbitrary three-manifolds seem to be out of reach for the moment.

Question. Are the inequalities (2) sufficient to determine the tunnel number for every link? In other words is it true that

$$t(L) + 1 = \max_{\rho} \Big(\frac{1}{q} \, \mu_R\big(H_1(C_L, \rho)\big)\Big). \tag{4}$$

where ρ ranges over all right representations $\pi_1(C_L) \to GL(q, R)$?

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LABORATOIRE MATHÉMATIQUES JEAN LERAY UMR 6629, UNIVERSITÉ DE NANTES, FACULTÉ DES SCIENCES, 2, RUE DE LA HOUSSINIÈRE, 44072, NANTES, CEDEX

E-mail address: `pajitnov@math.univ-nantes.fr`